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# CONVERGENCE OF THE METHOD OF SMALL PARAMETER FOR WEAKLY CONTROLLABLE OPTIMAL SYSTEMS

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Validity of the method of small parameter [1] for approximate solution of a class of optimal control problems is proved, and its rate of convergence is estimated.

**1. Statement of the problem.** Let us consider the weakly controllable system

$$x^{\cdot} = f^{\circ}(x, t) + \varepsilon f^1(x, t, u), \quad \varepsilon \in [0, \delta], u \in U \tag{1.1}$$

where $x$ is an $n$-dimensional vector, $u$ is the $r$-dimensional control vector, $\varepsilon$ is a positive parameter, and $U$ is a compact set in the $r$-dimensional Euclidean space. The process is assumed to begin at the fixed point

$$x(t_0) = a \tag{1.2}$$

We call function $u(t)$ admissible control, if it is measurable in $u(t) \in U$ for all $t$. We denote by $x_u^{\varepsilon}(t)$ the solution of the Cauchy problem for Eq. (1.1) with initial condition (1.2) and fixed admissible control $u(t)$ and $\varepsilon \in (0, \delta]$ and by $T_u^{\varepsilon}$ the first instant of time at which trajectory $x_u^{\varepsilon}(t)$ reaches surface $g(x, t) = 0$, where $g(x, t)$ is some scalar function, i.e. $T_u^{\varepsilon}$ is the minimum solution of equation

$$g(x_u^{\varepsilon}(t), t) = 0, \quad t > t_0 \tag{1.3}$$

Let us formulate the following variational problem : for a given $\varepsilon \in (0, \delta]$ find function $u^{\varepsilon}(t)$ (the optimal solution) that would yield the minimum of functional

$$J_u^{\varepsilon} = F(x_u^{\varepsilon}(T_u^{\varepsilon}), T_u^{\varepsilon}) \tag{1.4}$$

for all possible admissible control functions $u(t)$. In this formula $F(x, t)$ is some scalar function.

The distinctive feature of this problem is in that system (1.1) becomes uncontrollable (i.e. independent of $u$), if parameter $\varepsilon$ vanishes.

Let us assume that an optimal control $u^{\varepsilon}(t)$ exists for all $\varepsilon \in (0, \delta]$. We denote by $x^{\varepsilon}(t)$ at $T^{\varepsilon}$ the optimal trajectory at the instant of completion of the optimal process. In that case the necessary conditions of optimality are defined as follows. There exists a vector function $p^{\varepsilon}(t)$ such that

$$p^{\varepsilon\cdot} = -\nabla H(x^{\varepsilon}(t), t, u^{\varepsilon}(t), p^{\varepsilon}(t), \varepsilon) \tag{1.5}$$

$$p^{\varepsilon}(T^{\varepsilon}) = \left(\frac{F'}{g'}\nabla g - \nabla F\right)\Big|_{x^{\varepsilon}(t),\, u^{\varepsilon}(t),\, t=T^{\varepsilon}} \tag{1.6}$$

$$(p^{\varepsilon}(t), f^1(x^{\varepsilon}(t), t, u^{\varepsilon}(t)) = \max_{u \in U}(p^{\varepsilon}(t), f^1(x^{\varepsilon}(t), t, u)), \quad t \in [t_0, T^{\varepsilon}] \tag{1.7}$$

where

$$H(x, t, u, p, \varepsilon) = (p, f^\circ(x, t)) + \varepsilon (p, f^1(x, t, u))$$
$$F' = F_t(x, t) + (\nabla F(x, t), f^\circ(x, t) + \varepsilon f^1(x, t, u))$$

Function $g'$ is defined similarly to $F'$. Obviously $g'$ and $F'$ are total derivatives of functions $g$ and $F$ along the trajectory of system (1.1).

The method of small parameter is applied as follows [1]. We formally set in (1.1) $\varepsilon = 0$, which yields the Cauchy problem: $x^\cdot = f^\circ(x, t)$ and $x(t_0) = a$. We denote its solution by $x^\circ(t)$ and determine the instant of time $T^\circ$ as the first root of equation $g(x^\circ(t), t) = 0$. We then set $\varepsilon = 0$ in the right-hand sides of formulas (1.5) and (1.6). As the result we have

$$p^\cdot = -A^*(t)\, p$$
$$p(T^\circ) = \left[\frac{F'(x^\circ(t), t)}{g'(x^\circ(t), t)} \nabla g(x^\circ(t), t) - \nabla F(x^\circ(t), t)\right]\Bigg|_{t=T^\circ} \qquad (1.8)$$

where $A(t) = \partial f^\circ(x^\circ(t), t) / \partial x$, is a matrix with components $\partial f_i^\circ(x^\circ(t), t) / \partial x_j$, and $A^*(t)$ is the transposed matrix.

We denote the solution of this Cauchy problem by $p^\circ(t)$, and then use the notation

$$h^\varepsilon(t, u) = (p^\varepsilon(t), f^1(x^\varepsilon(t), t, u)), \quad h^\circ(t, u) = (p^\circ(t), f^1(x^\circ(t)\ t, u))$$

It is fairly clear that $x^\varepsilon(t) = x^\circ(t) + O(\varepsilon)$, and $p^\varepsilon(t) = p^\circ(t) + O(\varepsilon)$. Hence $h^\varepsilon(t, u) = h^\circ(t, u) + O(\varepsilon)$. This makes it reasonable to seek the approximate optimal control $u^\varepsilon(t)$ as the solution of an equation of the form

$$h^\circ(t, u) = \max_{u \in U} h^\circ(t, u) \qquad (1.9)$$

We denote by $u^\circ(t)$ the function which yields the maximum of $h^\circ(t, u)$ in the set $U$. Evidently $u^\circ(t) \in U$ and, furthermore, it is possible to show that function $u^\circ(t)$ is measureable (the particular case of Filippov's lemma in [2]). Hence $u^\circ(t)$ is an admissible control. Function $u^\circ(t)$ that satisfies (1.9) may not be unique, and in such case we take an arbitrary function that satisifes Eq. (1.9).

We are faced with the problem of determining the relation of the formally derived control $u^\circ(t)$ to the optimal control $u^\varepsilon(t)$. If we take an arbitrary admissible control $u(t)$, then $J^\varepsilon_{u^\varepsilon} - J_u{}^\varepsilon = O(\varepsilon)$ uniformly with respect to $u(t)$, since $x_u{}^\varepsilon(t) - x^\circ(t) = O(\varepsilon)$. It is shown below that on some fairly general assumptions $J^\varepsilon_{u^\varepsilon} - J_{u^\circ}{}^\varepsilon = O(\varepsilon^2)$.

**2. Auxilliary statements.** Let the following conditions be satisfied:

1) functions $f^\circ(x, t)$ and $f^1(x, t, u)$ are twice continuously differentiable with respect to $x$ and continuous with respect to $(t, u)$;

2) function $g(x, t)$ is twice continuously differentiable with respect to $(x, t)$, and function $\varphi^\circ(t) \equiv g(x^\circ(t), t)$ vanishes at instant $T^\circ > t_0$, while $\varphi^\circ(t) \neq 0$ for $t \in [t_0, T^\circ)$ and

$$\varphi^{\circ\cdot}(T^\circ) \neq 0 \qquad (2.1)$$

3) There exists such constant $b > 0$ that for all $\varepsilon \in [0, \delta]$ and all admissible $u(t)$ and any $t \in [t_0, T^*]$, where $T^*$ is a certain instant of time greater than $T^\circ$, the inequality

$$|x_u{}^\varepsilon(t)| \leqslant b \qquad (2.2)$$

is valid.

4) function $F(x, t)$ is twice continuously differentiable, and

5) an optimal control $u^\varepsilon$ exists for all $\varepsilon \in (0, \delta]$ . Under conditions 1) — 5) the following lemmas are satisfied.

Lemma 1. For all $\varepsilon \in [0, \delta]$ and any admissible $u(t)$

$$x_u^\varepsilon(t) = x^\circ(t) + \varepsilon x_u^1(t) + O(\varepsilon^2) \tag{2.3}$$

where $x_u^1(t)$ is the solution of equation

$$x_u^{1\cdot} = A(t) x_u^1 + f^1(x^\circ(t), t, u(t)), \quad x_u^1(t_0) = 0 \tag{2.4}$$

Estimate (2.3) is to be understood as proving the existence of a $\beta > 0$ independent of $\varepsilon \in [0, \delta]$, of the admissible control $u(t)$ and of $t \in [t_0, T^*]$, and such that ($\beta$ is a const.)

$$|x_u^\varepsilon(t) - x^\circ(t) - \varepsilon x_u^1(t)| \leqslant \varepsilon^2\beta, \quad t \in [t_0, T^*]$$

The proof of Lemma 1 is obtained by applying conventional reasoning used for proving expansions of solutions of regularly perturbed differential equations in series in a small parameter. The double continuous differentiability of functions $f^\circ$ and $f^1$ with respect to $x$ makes possible the expansion of $x_u^\varepsilon(t)$ to within $\varepsilon^2$, and Eq.(2.4) for $x_u^1(t)$ is obtained by obvious means, while the uniformity of estimate (2.3) with respect to admissible $u(t)$ (i.e. the independence of constant $\beta$ of the selection of $u(t)$) is established using the uniform boundedness of trajectories of Eq. (1.1) in $[t_0, T^*]$, i.e. using the inequality (2.2).

Lemma 2. There exists number $\delta^*$, $0 < \delta^* \leqslant \delta$, such that for all $\varepsilon \in [0, \delta^*]$ and any admissible $u(t)$ there exists an instant of time $T_u^\varepsilon$ at which the trajectory $x_u^\varepsilon(t)$ reaches surface $g(x, t) = 0$ and $T_u^\varepsilon \leqslant T^*$.

The meaning of this lemma is fairly clear. Since all trajectories of Eq. (1.1) lie in the $\varepsilon$-neighborhood of trajectory $x^\circ(t)$ which enters "fairly normally" the terminal surface at instant of time $T^\circ$ (by virtue of inequality (2.1)), hence for fairly small $\varepsilon$ all trajectories also reach the terminal surface at instant of time $T_u^\varepsilon$ which differs from $T^\circ$ by a quantity of order $\varepsilon$.

Lemma 3. For any $\varepsilon \in [0, \delta^*]$ and all admissible $u(t)$

$$T_u^\varepsilon = T^\circ + \varepsilon T_u^1 + O(\varepsilon^2) \tag{2.5}$$

uniformly with respect to $u(t)$, where

$$T_u^1 = -(\nabla g(x^\circ(T^\circ), T^\circ), x_u^1(T^\circ)) / \varphi^{\cdot\circ}(T^\circ) \tag{2.6}$$

Proof. Using estimate (2.3) and condition 2) we obtain

$$\varphi_u^\varepsilon(t) \equiv g(x_u^\varepsilon(t), t) = \varphi^\circ(t) + \varepsilon(\nabla g(x^\circ(t), t), x_u^1(t)) + O(\varepsilon^2)$$

$$\varphi_u^\varepsilon(T_u^\varepsilon) = 0 = \varphi^\circ(T_u^\varepsilon) + \varepsilon(\nabla g(x^\circ(T_u^\varepsilon), T_u^\varepsilon), x_u^1(T_u^\varepsilon)) + O(\varepsilon^2)$$

We seek $T_u^\varepsilon$ of the form $T_u^\varepsilon = T^\circ + \varepsilon T_u^1 + O(\varepsilon^2)$. Formula (2.6) is then readily obtainable. The uniformity of estimate (2.5) is proved using the continuous differentiability of function $\varphi^\circ(t)$ and condition (2.2).

Lemma 4. For any $\varepsilon \in [0, \delta^*]$ and all admissible $u(t)$

$$J_u^{\varepsilon} = F(x^\circ(T^\circ), T^\circ) - \varepsilon(p^\circ(T^\circ), x_u^1(T^\circ)) + O(\varepsilon^2) \qquad (2.7)$$

uniformly with respect to $u(t)$ and $p^\circ(T^\circ)$ determined by the right-hand side of the second formula (1.8).

Proof. Using estimate (2.3) and condition 4) we obtain

$$J_u^{\varepsilon} = F(x^\circ(T^\circ), T^\circ) + \varepsilon[F'(x^\circ(T^\circ) T^\circ) T_u^1 + (\nabla F(x^\circ(T^\circ), T^\circ), x_u^1(T^\circ))] + O(\varepsilon^2)$$

and the substitution of $T_u^1$ from (2.6) into this formulas yields (2.7)

Lemma 5. For all $\varepsilon \in (0, \delta^*]$

$$p^\varepsilon(t) = p^\circ(t) + O(\varepsilon) \qquad (2.8)$$

uniformly with respect to $t \in [t_0, T^*]$

The proof of this lemma directly follows from the double continuous differentiability of functions $F$, $g$, $f^\circ$, and $f^1$ with respect to $x$ and estimates (2.3) and (2.5).

Using the notation $T^{\circ\varepsilon} = \min\{T^\circ, T^\varepsilon\}$ we obviously have $T^\circ - T^{\circ\varepsilon} = O(\varepsilon)$.

Lemma 6. There exists a constant $\varkappa > 0$ such that

$$0 \leqslant h^\circ(t, u^\circ(t)) - h^\circ(t, u^\varepsilon(t)) \leqslant 2\varepsilon\varkappa, \quad \varepsilon \in (0, \delta^*], \quad t \in [t_0, T^{\circ\varepsilon}] \qquad (2.9)$$

Proof. The smoothness of function $f^1$ and estimates (2.3), (2.5), and (2.8) imply the existence of some constant $\varkappa > 0$ such that

$$|h^\varepsilon(t, u) - h^\circ(t, u)| \leqslant \varepsilon\varkappa, \quad t \in [t_0, T^{\circ\varepsilon}], \quad \varepsilon \in (0, \delta^*], \quad u \in U$$

which yields

$$h^\varepsilon(t, u^\varepsilon(t)) - h^\circ(t, u^\varepsilon(t)) \leqslant \varepsilon\varkappa, \quad h^\circ(t, u^\circ(t)) - h^\varepsilon(t, u^\circ(t)) \leqslant \varepsilon\varkappa$$

Furthermore by virtue of (1.7) $h^\varepsilon(t, u^\circ(t)) - h^\varepsilon(t, u^\varepsilon(t)) \leqslant 0$. Adding the last three inequalities we obtain

$$h^\circ(t, u^\circ(t)) - h^\circ(t, u^\varepsilon(t)) \leqslant 2\varepsilon\varkappa.$$

and since $u^\circ(t)$ is the solution of Eq. (1.9), this yields estimate (2.9).

**3. The basic theorem.** When conditions 1) – 5) are satisfied, then

$$J_{u^\varepsilon}^{\varepsilon} - J_{u^\circ}^{\varepsilon} = O(\varepsilon^2), \quad \varepsilon \in (0, \delta^*] \qquad (3.1)$$

Proof. From (2.7) using the notation $\Delta x^1(t) = x_{u^\circ}^1(t) - x_{u^\varepsilon}^1(t)$ we have

$$J_{u^\varepsilon}^{\varepsilon} - J_{u^\circ}^{\varepsilon} = \varepsilon(p^\circ(T^\circ), \Delta x^1(T^\circ)) + O(\varepsilon^2) = \varepsilon \int_{t_0}^{T^\circ} [(p^{\circ\cdot}(t), \Delta x^1(t)) + (p^\circ(t), \Delta x^{1\cdot}(t))]\, dt + O(\varepsilon^2)$$

Substituting in this formula

$$p^{\circ\cdot} = -A^* p^\circ, \; \Delta x^{1\cdot} = A\Delta x^1 + f^1(x^\circ(t), t, u^\circ(t)) - f^1(x^\circ(t), t, u^\varepsilon(t))$$

we obtain the expression

$$J^{\varepsilon}_{u^{\varepsilon}} - J^{\varepsilon}_{u^{\circ}} = \varepsilon \int_{t_0}^{T^{\circ\varepsilon}} [h^{\circ}(t, u^{\circ}(t)) - h^{\circ}(t, u^{\varepsilon}(t))] dt + \varepsilon \int_{T^{\circ\varepsilon}}^{T^{\circ}} [h^{\circ}(t, u^{\circ}(t)) - h^{\circ}(t, u^{\varepsilon}(t))] dt + O(\varepsilon^2)$$

Since in the last formula the integrand of the first integral is by Lemma 6 of order $\varepsilon$, while that of the second integral is necessarily bounded and the integration interval is of order $\varepsilon$, hence the whole right-hand side of that formula is of order $\varepsilon^2$. The theorem is proved.

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